Volume of water in tub = W
Area of Tub Base = A_t = \( l_1 l_2 \)
Depth of water in tub = h = W/A_t
Drain Velocity = \( v = (2gh)^{0.5} \)
Drain Discharge = \( A_d v \)

Figure 1. A simple sketch of the water tub system, consisting of a faucet, a drain, and tub that contains water. The faucet flow rate is independently controlled, but the rate of flow through the drain is a function of the water depth and the area of the drain opening.

Numerical Values & Expressions:
INIT bath tub = 1 (m^3)
faucet = 1 (m^3/sec)
drain = drain velocity*drain area (m^3/sec)
drain area = 0.05 (m^2)
area of tub base = 0.5 (m^2)
depth of water = water tub/area of tub base (m)
gravity = 9.8 (m/sec^2)
drain velocity = \( \text{SQRT}(2 \times \text{gravity} \times \text{depth of water}) \) (m/sec)

Figure 2. The bath tub system as represented in STELLA. The reservoir, flows, and converters all have numerical values associated with them; hidden in this view, they can be seen and changed by double-clicking on each symbol. If no numerical value or expression is associated with a symbol, the program shows a question mark to prompt the modeler to enter a value or equation. The numerical values and expressions used in this model are given in the box. INIT stands for the initial amount of water in the tub; comments enclosed in { } brackets are used to help keep track of units; note that the time units here are seconds.
Figure 3. Results of running the bathtub model for 100 seconds using the initial conditions given in Figure 2. After about 75 seconds, the system is in a steady state.

Note that the vertical scale is the same for the drain and faucet curves, but is different for the water tub curve.
Figure 4. Results of running the bathtub model 5 different times, varying the initial amount of water in the tub. In each case, the system returns to its steady state in about 75 seconds, with a response time of around 20 seconds.

Each curve represents a different model run, with differing initial amounts of water in the tub.

Volume of Water in Tub (m$^3$)

Response Time = time needed to accomplish 63% of change required to get to steady state
Positive Feedback Leads to Runaway Behavior

Figure 5. Modified bathtub system to illustrate a positive feedback mechanism. In this case, the faucet is defined such that its rate of inflow increases as the amount in the tub increases, while the drain is defined as a constant (physically unreal, of course). The model was run 3 times with different initial amounts of water; initial values greater than 5.0 trigger the positive feedback mechanism, leading to runaway behavior that follows an exponential curve. Comparing the reservoir values at 50 seconds and 100 seconds shows the impressive increases that result from exponential growth — 38 million m$^3$ of water in the case where the initial value is 5.2 m$^3$.\n
Note the dramatic change in the vertical axis for this segment from 50 to 100 seconds.
Two-Tub System

Figure 6. A system with two bathtubs connected by a drain illustrates the concept of lag time. Both tubs start out with the same amount of water, and the drains have identical rate constants. The faucet is defined as a graphical function of time, starting at a constant rate of 1 m³/s, then jumping up to 4 m³/s at time = 2, then returning to 1 m³/s for the duration of the time. Water tub 1 peaks at a value of 12.85 m³ at time = 3, so this reservoir has a lag time of 1 second; tub 2 peaks much later, at time = 12, so its lag time is 10 seconds behind the faucet peak, and 9 seconds behind the "upstream" reservoir.

<table>
<thead>
<tr>
<th>Water Tub 1</th>
<th>Water Tub 2</th>
<th>Faucet Rate (m³/sec)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>2.50</td>
<td>2.50</td>
<td>2.50</td>
</tr>
<tr>
<td>5.00</td>
<td>5.00</td>
<td>5.00</td>
</tr>
<tr>
<td>10.00</td>
<td>10.00</td>
<td>10.00</td>
</tr>
</tbody>
</table>

Lag Times Revealed in Response to a Perturbation
Figure 7. Three variations of the bathtub model, with varying degrees of complexity. The simplest model has a constant drain, while in the more realistic model, the drain flow rate is calculated using Torricelli’s Law. The model of intermediate complexity just represents the drain flow as a rate constant multiplied by the amount of water in the tub. Comparing the models shows virtually no difference between the intermediate complexity model and the more realistic one. The intermediate model is thus an acceptably simple model—not so simple that it fails to capture the essence of the system.
Increased Connectedness Leads to Shorter Response Times

Figure 8. Here, two reservoirs with typical draining flow involving rate constants $k_1$ and $k_2$ are shown in a connected and a disconnected state. The connected system shows a shorter response time, given by $\frac{1}{k_1 + k_2}$. The isolated reservoirs, on the other hand, show response times $\frac{1}{k_1} = 10$ and $\frac{1}{k_2} = 200$. This comparison illustrates how increased connectedness leads to shorter response times.
All flows are defined as the product of the reservoir, \( M \), they drain times the rate constant, \( k \).

A 4-reservoir system has some difficulty reaching its steady state; many reservoirs "overshoot" their steady-state values and the effective response time of the system is greater than might be expected.

If we disassemble the system below into four separate systems (as in Fig. 8b) consisting of reservoirs with outflows and no inflows, the response times of each subsystem are:

\[
\begin{align*}
  t_{M1} &= \frac{1}{k_{12}} = 2.5 \text{ yrs} \\
  t_{M2} &= \frac{1}{k_{23}} = 5 \text{ yrs} \\
  t_{M3} &= \frac{1}{k_{34}} = 1.67 \text{ yrs} \\
  t_{M4} &= \frac{1}{k_{41}} = 3.33 \text{ yrs}
\end{align*}
\]

The modified system, with more flows, still exhibits the "overshoot" tendency, but reaches its steady state faster than the model with fewer connections; its response time is shorter.

**Figure 9. Two versions of a larger system.** The upper system shows that the response time of larger systems is not as easily predicted as in the case of smaller, two-reservoir systems. In larger systems, some reservoirs may overshoot their eventual steady state levels, thus obliterating their eventual arrival at the steady state. The lower version is identical to the upper system, with the addition of two new flows; the result is an overall decrease in the response time.
Figure 11. The results of running the STELLA model of the global carbon cycle with the history of fossil fuel emissions and land-use changes (from Table 1), shown below, compare fairly well with the observed record of atmospheric CO2 build-up over the same time period. The model is thus validated (in a relatively simple, first-order sense), meaning that the model's results are potentially meaningful in a quantitative sense.