Lee Silverman Geo 161 Assignment due 10/14/93

a) We have two equations; one for the mass of the earth and one for the moment of inertia:

$I_e = I_m + I_c$

$.3308\ M_e a^2 = \frac{8}{15}\pi \rho_m r_e^5 + \frac{8}{15}\pi \rho_\Delta r_c^5$ where $\rho_m + \rho_\Delta = \rho_c$

$8.021\times10^{44}\ g\,cm^2 = 5.799\times10^{44} + \frac{8}{15}\pi \rho_\Delta r_c^5$

$1.326\times10^{44} = \rho_\Delta r_c^5$

$M_e = M_m + M_c$

$5.976\times10^{27}\,g = \frac{4}{3}\pi \rho_m r_e^3 + \frac{4}{3}\pi \rho_\Delta r_c^3$

$5.976\times10^{27}\,g = 3.573\times10^{27} + \frac{4}{3}\rho_\Delta r_c^3$

$5.737\times10^{26} = \rho_\Delta r_c^3$

Equating the two equations to ρ_Δ, we have

$$\frac{1.326\times10^{44}}{r_c^5} = \frac{5.737\times10^{26}}{r_c^3}$$

$$2.311\times10^{17} = r_c^2$$

$$\boxed{4.808\times10^8\,cm = r_c}$$

Plugging back in, $\frac{1.326\times10^{44}}{(4.808\times10^8)^5} = 5.161\ \frac{g}{cm^3} = \rho_\Delta$

$\rho_c = \rho_m + \rho_\Delta = 3.3 + 5.2 = \boxed{8.5\frac{g}{cm^3} = \rho_c}$

— b We're going to end up with two equations in three unknowns. Plugging in everything we know:

$$5.976\times10^{27}\ gm = 1.083\times10^{27}\rho_m + 4.189\,(\rho_D r_c^3)$$

$$8.021\times10^{44} = 1.757\times10^{44}\rho_m + 1.676\,(\rho_D r_c^5)$$

or

$$\text{I} \quad 5.518 = \rho_m + 3.868\times10^{-27}(\rho_D r_c^3)$$

$$\text{II} \quad 4.565 = \rho_m + 9.539\times10^{-45}(\rho_D r_c^5)$$

The radius of the core must be less than 10^9 cm. If the radius of the core is that large, then according to the right hand side of both equations, ρ_D must be of order 1. However, if r_c is of order 10^9, then for the right hand term in I to make a contribution, ρ_D must be of order 10^3. If that is so, then the right hand term in II is of order 10^{-3}. From this brief explanation, it should be clear that for any reasonable radius, the second term of II can be neglected as zero, but the second term of I is not necessarily neglectable, if ρ_D is made large enough. Setting the second term in II to be zero, we see:

$$5.518 = \rho_m + 3.868\times10^{-27}(\rho_D r_c^3)$$

$$4.565 = \rho_m$$

Hence, $\boxed{\text{The maximum density of the mantle must be } 4.565\ \frac{g}{cm^3}}$

Plugging into I, we obtain the relationship

$$\rho_D r_c^3 = 2.464\times10^{26}$$

But we cannot isolate either ρ_D or r_c with just these equations

c) plugging into I and II in b, we see:

$$5.518 = \rho_m + 1.658\times10^{-1}\rho_\Delta$$

$$4.565 = \rho_m + 5.010\times10^{-2}\rho_\Delta$$

Solving for ρ_m in the first equation and plugging into the second:

$$4.565 = 5.518 - 1.658\times10^{-1}\rho_\Delta + 5.010\times10^{-2}\rho_\Delta$$

$$-9.530\times10^{-1} = -1.157\times10^{-1}\rho_\Delta \Rightarrow \rho_\Delta = 8.237 \text{ g/cm}^3$$

plugging back into the first equation:

$$5.518 = \rho_m + (1.658\times10^{-1})(8.237) \Rightarrow \boxed{\rho_m = 4.152 \text{ g/cm}^3}$$

which implies $\rho_c = \rho_m + \rho_\Delta \Rightarrow \boxed{\rho_c = 12.389 \text{ g/cm}^3}$

d) The moment of inertia is given by

$$I = \int \rho \cdot r^2\, dV$$ over the volume of the earth.

Choosing the same coordinate system that we used in class,

$$I = \int_0^a\int_0^\pi\int_0^{2\pi} \rho \cdot R^2 (r^2 \sin\theta \, d\phi\, d\theta\, dr)$$ where R is the distance from the rotation axis

Plugging in $\rho = c - br$ and $R = r\sin\theta$, we have,

$$I = \int_0^a \int_0^\pi \int_0^{2\pi} cr^4\sin^3\theta - br^5\sin^3\theta \, d\phi\, d\theta\, dr$$

$$= 2\pi \int_0^a \int_0^\pi cr^4\sin^3\theta - br^5\sin^3\theta \, d\theta\, dr$$

Knowing that $\int \sin^3\theta\, d\theta = \frac{1}{3}\cos\theta(\sin^2\theta + 2)$, we have

$$I = 2\pi \int_0^a cr^4 - br^5 \left(\int_0^\pi \sin^3\theta \, d\theta \right) dr$$

$$= 2\pi \int_0^a cr^4 - br^5 \left[\frac{-1}{3}\cos\theta \left(\sin^2\theta + 2 \right) \right]_0^\pi$$

$$= 2\pi \int_0^a cr^4 - br^5 \left[\frac{1}{3}(2) - \left(\frac{-1}{3}\right)(2) = \frac{4}{3} \right] dr$$

$$= \frac{8}{3}\pi \int_0^a cr^4 - br^5 \, dr$$

$$= \frac{8}{3}\pi \left[\frac{1}{5}ca^5 - \frac{1}{6}ba^6 \right] = \frac{8}{3}\pi a^5 \left(\frac{c}{5} - \frac{b}{6}a \right)$$

to find the mass, we simply take

$$M = \int \rho \, dV$$ over the volume

Choosing the same coordinate system,

$$M = \int_0^a \int_0^\pi \int_0^{2\pi} \rho(r) \, r^2 \sin\theta \, d\phi \, d\theta \, dr$$

$$M = 2\pi \int_0^a \int_0^\pi (cr^2 - br^3)\sin\theta \, d\theta \, dr$$

$$= 4\pi \int_0^a cr^2 - br^3 \, dr$$

$$= 4\pi \left[\frac{1}{3}ca^3 - \frac{1}{4}ba^4 \right] = 4\pi a^3 \left(\frac{c}{3} - \frac{b}{4}a \right)$$

e What we really have here is a sphere of radius a composed entirely of material ρ_m, where ρ_m varies with radius as $\rho_m(r) = c - br$. Superimposed on this sphere is a sphere of constant density ρ_Δ, with radius 3500km. Thus, in the core, the density is a uniform factor of ρ_Δ above what it would be for mantle material, but the density decays at the same rate as mantle density decays.

Equations:

$$c_m - b_m a = 3.3$$

$$M_m + M_c = M_E$$

$$I_m + I_c = I_E$$

Taking M_m and I_m from part d, and using M_c and I_c the mass and moment of a uniform sphere with density ρ_Δ, we have

1a $c_m - b_m(6.37\times10^8) = 3.3$

1b $5.967\times10^{27} = 4\pi(6.37\times10^8)^3\left(\frac{c_m}{3} - \frac{b_m}{4}(6.37\times10^8)\right) + \frac{4}{3}\pi(3.5\times10^8)^3\rho_\Delta$

1c $8.021\times10^{44} = \frac{8}{3}\pi(6.37\times10^8)^5\left(\frac{c_m}{5} - \frac{b_m}{6}(6.37\times10^8)\right) + \frac{8}{15}\pi\rho_\Delta(3.5\times10^8)^5$

Simplifying:

2a $c_m = 3.3 + 6.37\times10^4 b_m$

2b $5.518 = c_m - 4.79\times10^8 b_m + 1.658\times10^{-1}\rho_\Delta$

2c $4.565 = c_m - 5.31\times10^8 b_m + 5.01\times10^{-2}\rho_\Delta$

Plugging 2a into 2b and 2c

3a $5.518 = 3.3 + 6.37\times10^8 b_m - 4.78\times10^8 b_m + 1.658\times10^{-1}\rho_\Delta$

3b $4.565 = 3.3 + 6.37\times10^8 b_m - 5.31\times10^8 b_m + 5.01\times10^{-2}\rho_\Delta$

Or, Simplifying,

4a $2.218 = 1.59\times10^{8}\, b_m + 1.658\times10^{-1} \rho_D$

4b $1.265 = 1.060\times10^{8}\, b_m + 5.01\times10^{-2} \rho_D$

Solving 4b for ρ_D

5 $2.525\times10^{1} = 2.116\times10^{9}\, b_m + \rho_D \Rightarrow \rho_D = 25.25 - 2.116\times10^{9}\, b_m$

Plugging 5 into 4a

$$6 \begin{cases} 2.218 = 1.59\times10^{8}\, b_m + (1.658\times10^{-1})(25.25) - (1.658\times10^{-1})(2.116\times10^{9})\, b_m \\ -1.968 = -1.918\times10^{8}\, b_m \quad \Rightarrow \boxed{b_m = 1.026\times10^{-8}} \end{cases}$$

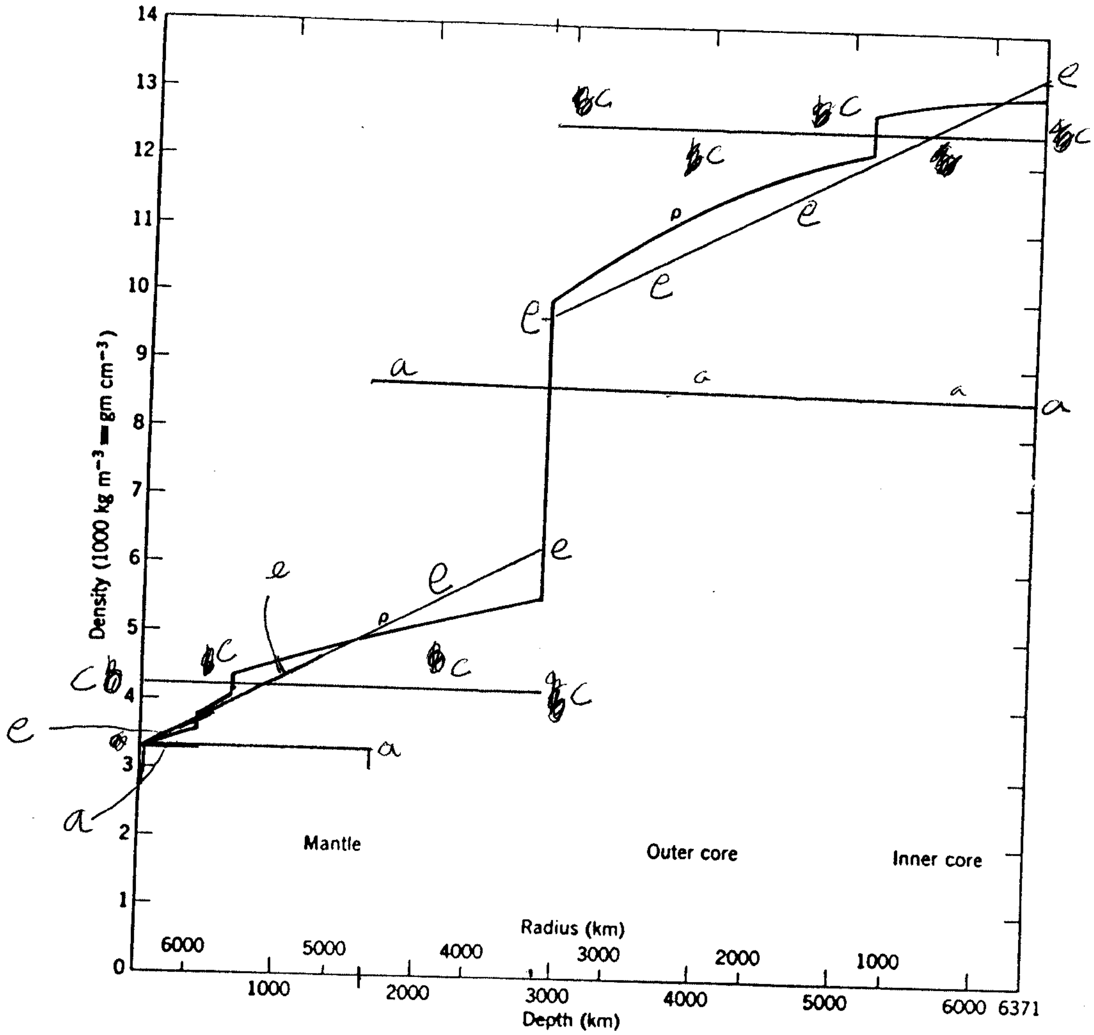
Plugging 6 into 4b,

$$7 \begin{cases} 1.265 = (1.060\times10^{8})(1.026\times10^{-8}) + 5.01\times10^{-2} \rho_D \\ 1.774\times10^{-1} = 5.01\times10^{-2} \rho_D \ ; \quad \boxed{\rho_D = 3.542} \end{cases}$$

Plugging 6 into 2a

9 $C_m = 3.3 + (6.37\times10^{8})(1.192\times10^{-8}) = C_m = \boxed{9.836}$

f) Models a, c, and e represent increasingly good fits to the seismically determined density structure of the earth. Model a is not very good - it consistently underestimates density, though the density contrast at the CMB is almost right. Model c is a little better, estimating about the average mantle density but overestimating core density. The density contrast at the CMB is too small. Model e is the best, modeling the mantle very well, while not quite so accurate in the core. Allowing a different density gradient in the core would have helped here. The density contrast at the CMB is too small in this model.



Density profile of Earth model by Dziewonski et al. (1975) (solid line)

model a)

$$r = \begin{cases} 0-4808 \text{ km}: & \rho = 8.5 \text{ g/cm}^3 \\ 4808-6371 \text{ km}: & \rho = 3.3 \text{ g/cm}^3 \end{cases}$$

model c)

$$r = \begin{cases} 0-3500 \text{ km}: & \rho = 12.5 \text{ g/cm}^3 \\ 3500-6371 \text{ km}: & \rho = 4.15 \text{ g/cm}^3 \end{cases}$$

model e)

$$r = \begin{cases} 0-3500 \text{ km}: & \rho = 13.38 - 1.026\times10^{-3}\, r \\ 3500-6371 \text{ km}: & \rho = 9.836 - 1.026\times10^{-3}\, r \end{cases}$$

r in km