Comments This is one of my favorite problem assignments in our Solid Earth Geophysics class, typically taken by junior and senior concentrators and by first-year graduate students. I encourage students to work together on it and/or consult with me (or the TA on the rare occasions that this course has had a TA). There are often planetary geology students taking this course, so it is useful for them to see how models of interiors of planets can be constrained. I preface the problem with notes of some of what I review in class before making the assignment and provide an answer copied from one of the students. This particular student wrote out everything in detail, so the answer is much longer than typical - also worked in cgs, rather than mks units, but otherwise answer is correct.

The homework problem requires the students to use the physical concept of moment of inertia; to plot models of density structure that reinforce knowledge of the density stratification of the earth; to do a triple integral in spherical coordinates, following the template above given in class; and to solve sets of simultaneous equations in two or three unknowns, using concepts from linear algebra.

Although these equations can be solved entirely by hand with use of a hand calculator, I encourage the students to try solving them with a computer routine, such as MATLAB. As I warn the students, solving with computer routines usually fails initially, because the coefficients differ in size by many orders of magnitude. Thus, this also can become a lesson in round-off errors or machine accuracy - further lessons in linear algebra result in showing them how to renormalize the equations or variables to restore accuracy.

Don Forsyth

Moment of Inertia Review

moment of inertia, I, is the rotational analog to mass

Linear momentum $p = mv$
where m = mass, v = velocity

Angular momentum $L = I\omega$
where I = moment of inertia, ω = angular velocity

Kinetic energy $= \frac{1}{2} mv^2$

Rotational kinetic energy $T = \frac{1}{2} I\omega^2$

Force $F = \frac{dp}{dt} = m\frac{dv}{dt}$

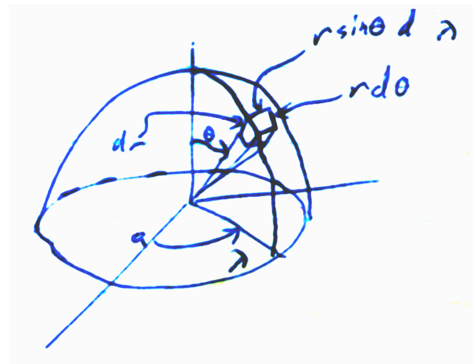
Torque $N = \frac{dL}{dt} = I\frac{d\omega}{dt}$

These equations valid for linear motion or rotation about fixed axis. In general, velocity, angular velocity, momentum, angular momentum, force and torque are all 3-D vectors.

Moment of inertia of a solid body about an axis

$I = \iiint \rho R^2 dV$ where ρ is density, R is distance from rotation axis, and dV is volume element.

Example: Moment of inertia of uniform sphere



φ latitude, θ colatitude, λ longitude, r distance from center

In spherical coordinates, volume element
$dV = r^2 \sin\theta \, dr \, d\theta \, d\lambda$

Radius of sphere = a $\qquad$ $R = r \sin\theta$

$$I = \int_{r=0}^{a} \int_{\theta=0}^{\pi} \int_{\lambda=0}^{2\pi} \rho (r \sin\theta)^2 r^2 \sin\theta \, dr \, d\theta \, d\lambda$$

In general ρ (r, θ, λ). In spherically symmetric body, density is function of radius only, ρ(r), so

$$I = 2\pi \int_0^a \rho r^4 \int_0^\pi \sin^3\theta \, d\theta \, dr \qquad \text{but} \int_0^\pi \sin^3\theta \, d\theta = \int_0^\pi (1-\cos^2\theta)\sin\theta \, d\theta = \left[\frac{\cos^3\theta}{3} - \cos\theta\right]_0^\pi = \frac{4}{3}$$

so $I = \frac{8\pi}{3} \int_0^a \rho r^4 \, dr$. If ρ is a constant, then $I = \frac{8\pi}{15} \rho a^5$.

A similar integral can be performed to find the mass, M, of the sphere
$M = \iiint \rho r^2 \sin\theta \, dr \, d\theta \, d\lambda = \frac{4}{3}\pi \rho a^3$, if density is constant. Combining these two expressions, we find that the moment of inertia of a solid, uniform sphere is

$$I = \frac{2}{5} M a^2 = 0.4 M a^2$$

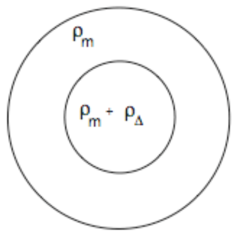
As is shown in the next week of lectures, the moment of inertia of the earth, instead of being 0.4 Ma^2, is 0.331 Ma^2. Given the definition of moment of inertia, what does this imply about density distribution in the earth as a function of radius?

The moment of inertia and the mass are the two primary constraints we have on the internal constitution of the planets, including the earth. The homework assignment demonstrates how these observations can constrain models of the planet Earth, especially in conjunction with one or two additional pieces of information.

Geo 161 Due Thursday, September 12, 2002

Use mass and moment of inertia of the earth to constrain the density distribution of the earth.

a. Assume the earth is radially symmetric and composed of 2 layers - a uniform outer mantle and a uniform core. Neglect the earth's crust. Rocks we think are representative of the earth's mantle have a density of about 3300 kg/m^3 at the earth's surface. Assuming that this is the density of the mantle, ρ_m, compute the radius and density of the core. You will need the mass of the earth, M_e = 5.976×10^{24} kg and moment of inertia, $I_e = .3308\ M_e\ a^2$. Radius of earth, $a = 6.37 \times 10^6$ m. Simplest approach is to assume a uniform sphere with density ρ_m and the radius of the earth, then consider radius of inner uniform sphere (core) with additional density, ρ_Δ, such that the density of the core $\rho_c = \rho_m + \rho_\Delta$. The sum of the effects of the two uniform spheres have to satisfy two constraints, the total mass and total moment of inertia, thus giving two equations in the two unknowns, r_{core} and ρ_Δ.



b. 3300 kg/m^3 is probably the minimum density of the mantle, since rocks are compressed with depth by increasing pressure. The result of part (a) then gives the maximum possible radius of the core. What is the maximum possible density of the outer shell consistent with the mass and moment of inertia constraints? (let $r_{core} \rightarrow 0$ and $\rho_\Delta \rightarrow \infty$). You will have to <u>think</u> here about how to satisfy both constraints simultaneously

$$M_e = \frac{4}{3}\pi\rho_m a^3 + \frac{4}{3}\pi\rho_\Delta r_{core}^3$$
$$I_e = \frac{8}{15}\pi\rho_m a^5 + \frac{8}{15}\pi\rho_\Delta r_{core}^5$$

c. From seismology, we know that the radius of the core is about 3500 km -- what are the densities of the core and mantle in this case?

d. What are the mass and moment of inertia in a sphere in which density increases linearly with increasing depth (i.e., $\rho(r) = c - br$)? (Evaluating integrals analytically is all that is required here, no numbers).

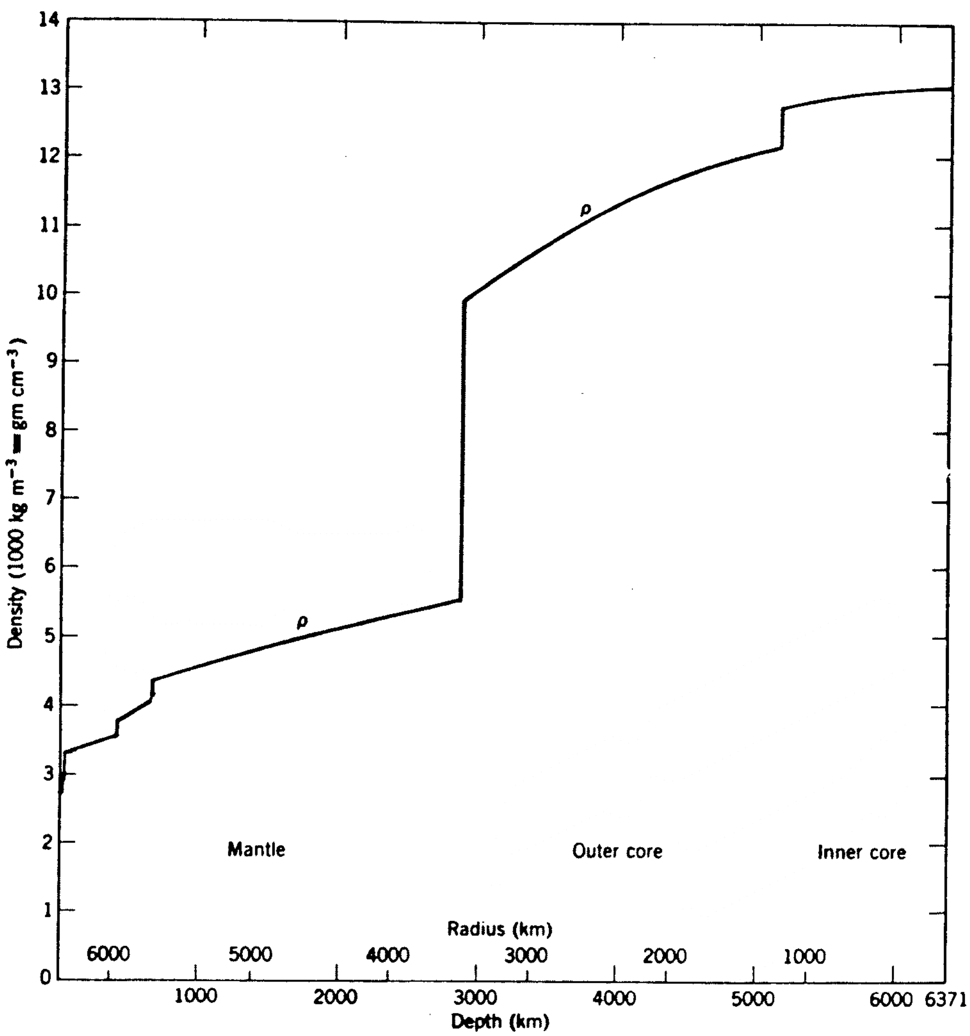
e. Assume the radius of the core = 3500 km. Assume the density in both the core and the mantle increase linearly with depth with the same gradient, and that the density at the surface ($r = a$) is 3300 kg/m^3. What is the profile of density in the core and the mantle? Use your analytical expression from part d for whole earth and add the extra effect of uniform extra core density, as in part a.

This part requires the solution of three equations in three unknowns, c, b_m, and ρ_Δ. Set up in form

$$c - b_m a = 3300$$
$$M_m + M_c = M_e$$
$$I_m + I_c = I_e$$

where M_m and I_m, the mass and moment of inertia of the whole earth not counting the extra effect of increased density in the core, are functions of b_m and c, and M_c and I_c, the mass and moment of inertia of the <u>extra</u> density of the core, are functions of ρ_Δ.

f. How do your models a, c, and e compare to detailed density profiles of the earth? (i.e., plot on diagram below and discuss briefly.)



Density profile of Earth model by Dziewonski et al. (1975) (solid line)

Lee Silverman Geo 161 Assignment due 10/14/93

a) We have two equations; one for the mass of the earth and one for the moment of inertia:

$I_e = I_m + I_c$

$.3308\ M_e a^2 = \frac{8}{15}\pi \rho_m r_e^5 + \frac{8}{15}\pi \rho_\Delta r_c^5$ where $\rho_m + \rho_\Delta = \rho_c$

$8.021\times10^{44}\ g\,cm^2 = 5.799\times10^{44} + \frac{8}{15}\pi \rho_\Delta r_c^5$

$1.326\times10^{44} = \rho_\Delta r_c^5$

$M_e = M_m + M_c$

$5.976\times10^{27}\ g = \frac{4}{3}\pi \rho_m r_e^3 + \frac{4}{3}\pi \rho_\Delta r_c^3$

$5.976\times10^{27}\ g = 3.573\times10^{27} + \frac{4}{3}\rho_\Delta r_c^3$

$5.737\times10^{26} = \rho_\Delta r_c^3$

Equating the two equations to ρ_Δ, we have

$$\frac{1.326\times10^{44}}{r_c^5} = \frac{5.737\times10^{26}}{r_c^3}$$

$$2.311\times10^{17} = r_c^2$$

$$\boxed{4.808\times10^8\ cm = r_c}$$

Plugging back in, $\frac{1.326\times10^{44}}{(4.808\times10^8)^5} = 5.161\ \frac{g}{cm^3} = \rho_\Delta$

$\rho_c = \rho_m + \rho_\Delta = 3.3 + 5.2 = \boxed{8.5\ \frac{g}{cm^3} = \rho_c}$

— b We're going to end up with two equations in three unknowns. Plugging in everything we know:

$$5.976\times10^{27}\ gm = 1.083\times10^{27}\rho_m + 4.189\,(\rho_\Delta r_c^3)$$

$$8.021\times10^{44} = 1.757\times10^{44}\rho_m + 1.676\,(\rho_\Delta r_c^5)$$

or

$$\text{I} \quad 5.518 = \rho_m + 3.868\times10^{-27}(\rho_\Delta r_c^3)$$

$$\text{II} \quad 4.565 = \rho_m + 9.539\times10^{-45}(\rho_\Delta r_c^5)$$

The radius of the core must be less than 10^9 cm. If the radius of the core is that large, then according to the right hand side of both equations, ρ_Δ must be of order 1. However, if r_c is of order 10^9, then for the right hand term in I to make a contribution, ρ_Δ must be of order 10^3. If that is so, then the right hand term in II is of order 10^{-3}. From this brief explanation, it should be clear that for any reasonable radius, the second term of II can be neglected as zero, but the second term of I is not necessarily neglectable, if ρ_Δ is made large enough. Setting the second term in II to be zero, we see:

$$5.518 = \rho_m + 3.868\times10^{-27}(\rho_\Delta r_c^3)$$

$$4.565 = \rho_m$$

Hence, $\boxed{\text{The maximum density of the mantle must be } 4.565\ \tfrac{g}{cm^3}}$

Plugging into I, we obtain the relationship

$$\rho_\Delta r_c^3 = 2.464\times10^{26}$$

But we cannot isolate either ρ_Δ or r_c with just these equations

c) plugging into I and II in b, we see:

$5.518 = \rho_m + 1.658\times10^{-1}\rho_\Delta$

$4.565 = \rho_m + 5.010\times10^{-2}\rho_\Delta$

Solving for ρ_m in the first equation and plugging into the second:

$4.565 = 5.518 - 1.658\times10^{-1}\rho_\Delta + 5.010\times10^{-2}\rho_\Delta$

$-9.530\times10^{-1} = -1.157\times10^{-1}\rho_\Delta \Rightarrow \rho_\Delta = 8.237\ \mathrm{g/cm^3}$

plugging back into the first equation:

$5.518 = \rho_m + (1.658\times10^{-1})(8.237) \Rightarrow \boxed{\rho_m = 4.152\ \mathrm{g/cm^3}}$

which implies $\rho_c = \rho_m + \rho_\Delta \Rightarrow \boxed{\rho_c = 12.389\ \mathrm{g/cm^3}}$

d) The moment of inertia is given by

$I = \int \rho \cdot r^2\, dV$ over the volume of the earth.

Choosing the same coordinate system that we used in class,

$$I = \int_0^a\int_0^\pi\int_0^{2\pi} \rho \cdot R^2 (r^2 \sin\theta\, d\lambda\, d\theta\, dr)$$ where R is the distance from the rotation axis

Plugging in $\rho = c - br$ and $R = r\sin\theta$, we have,

$$I = \int_0^a \int_0^\pi \int_0^{2\pi} cr^4 \sin^3\theta - br^5 \sin^3\theta\, d\lambda\, d\theta\, dr$$

$$= 2\pi \int_0^a \int_0^\pi cr^4 \sin^3\theta - br^5 \sin^3\theta\, d\theta\, dr$$

Knowing that $\int \sin^3\theta\, d\theta = \frac{1}{3}\cos\theta(\sin^2\theta + 2)$, we have

$$I = 2\pi \int_0^a cr^4 - br^5 \left(\int_0^\pi \sin^3\theta \, d\theta \right) dr$$

$$= 2\pi \int_0^a cr^4 - br^5 \left[\frac{-1}{3} \cos\theta \left(\sin^2\theta + 2 \right) \right]_0^\pi$$

$$= 2\pi \int_0^a cr^4 - br^5 \left[\frac{1}{3}(2) - \left(\frac{-1}{3}\right)(2) = \frac{4}{3} \right] dr$$

$$= \frac{8}{3}\pi \int_0^a cr^4 - br^5 \, dr$$

$$= \frac{8}{3}\pi \left[\frac{1}{5} ca^5 - \frac{1}{6} ba^6 \right] = \frac{8}{3}\pi a^5 \left(\frac{c}{5} - \frac{b}{6} a \right)$$

to find the mass, we simply take

$$M = \int \rho \, dV \quad \text{over the volume}$$

Choosing the same coordinate system,

$$M = \int_0^a \int_0^\pi \int_0^{2\pi} \rho(r) \, r^2 \sin\theta \, dz \, d\theta \, dr$$

$$M = 2\pi \int_0^a \int_0^\pi (cr^2 - br^3) \sin\theta \, d\theta \, dr$$

$$= 4\pi \int_0^a cr^2 - br^3 \, dr$$

$$= 4\pi \left[\frac{1}{3} ca^3 - \frac{1}{4} ba^4 \right] = 4\pi a^3 \left(\frac{c}{3} - \frac{b}{4} a \right)$$

e What we really have here is a sphere of radius a composed entirely of material ρ_m, where ρ_m varies with radius as $\rho_m(r) = c - br$. Superimposed on this sphere is a sphere of constant density ρ_D, with radius 3500 km. Thus, in the core, the density is a uniform factor of ρ_D above what it would be for mantle material, but the density decays at the same rate as mantle density decays.

Equations:

$$c_m - b_m a = 3.3$$

$$M_m + M_c = M_E$$

$$I_m + I_c = I_E$$

Taking M_m and I_m from part d, and using M_c and I_c the mass and moment of a uniform sphere with density ρ_D, we have

1a $c_m - b_m(6.37\times10^8) = 3.3$

1b $5.967\times10^{27} = 4\pi(6.37\times10^8)^3\left(\frac{c_m}{3} - \frac{b_m}{4}(6.37\times10^8)\right) + \frac{4}{3}\pi(3.5\times10^8)^3\rho_D$

1c $8.021\times10^{44} = \frac{8}{3}\pi(6.37\times10^8)^5\left(\frac{c_m}{5} - \frac{b_m}{6}(6.37\times10^8)\right) + \frac{8}{15}\pi\rho_D(3.5\times10^8)^5$

Simplifying:

2a $c_m = 3.3 + 6.37\times10^8 b_m$

2b $5.518 = c_m - 4.78\times10^8 b_m + 1.658\times10^{-1}\rho_D$

2c $4.565 = c_m - 5.31\times10^8 b_m + 5.01\times10^{-2}\rho_D$

Plugging 2a into 2b and 2c

3a $5.518 = 3.3 + 6.37\times10^8 b_m - 4.78\times10^8 b_m + 1.658\times10^{-1}\rho_D$

3b $4.565 = 3.3 + 6.37\times10^8 b_m - 5.31\times10^8 b_m + 5.01\times10^{-2}\rho_D$

Or, Simplifying,

4a $2.218 = 1.59\times10^{8}\, b_m + 1.658\times10^{-1} \rho_D$

4b $1.265 = 1.060\times10^{8}\, b_m + 5.01\times10^{-2} \rho_D$

Solving 4b for ρ_D

5 $2.525\times10^{1} = 2.116\times10^{9}\, b_m + \rho_D \Rightarrow \rho_D = 25.25 - 2.116\times10^{9}\, b_m$

Plugging 5 into 4a

6 $\begin{cases} 2.218 = 1.59\times10^{8}\, b_m + (1.658\times10^{-1})(25.25) - (1.658\times10^{-1})(2.116\times10^{9})\, b_m \\ -1.968 = -1.918\times10^{8}\, b_m \quad \Rightarrow \boxed{b_m = 1.026\times10^{-8}} \end{cases}$

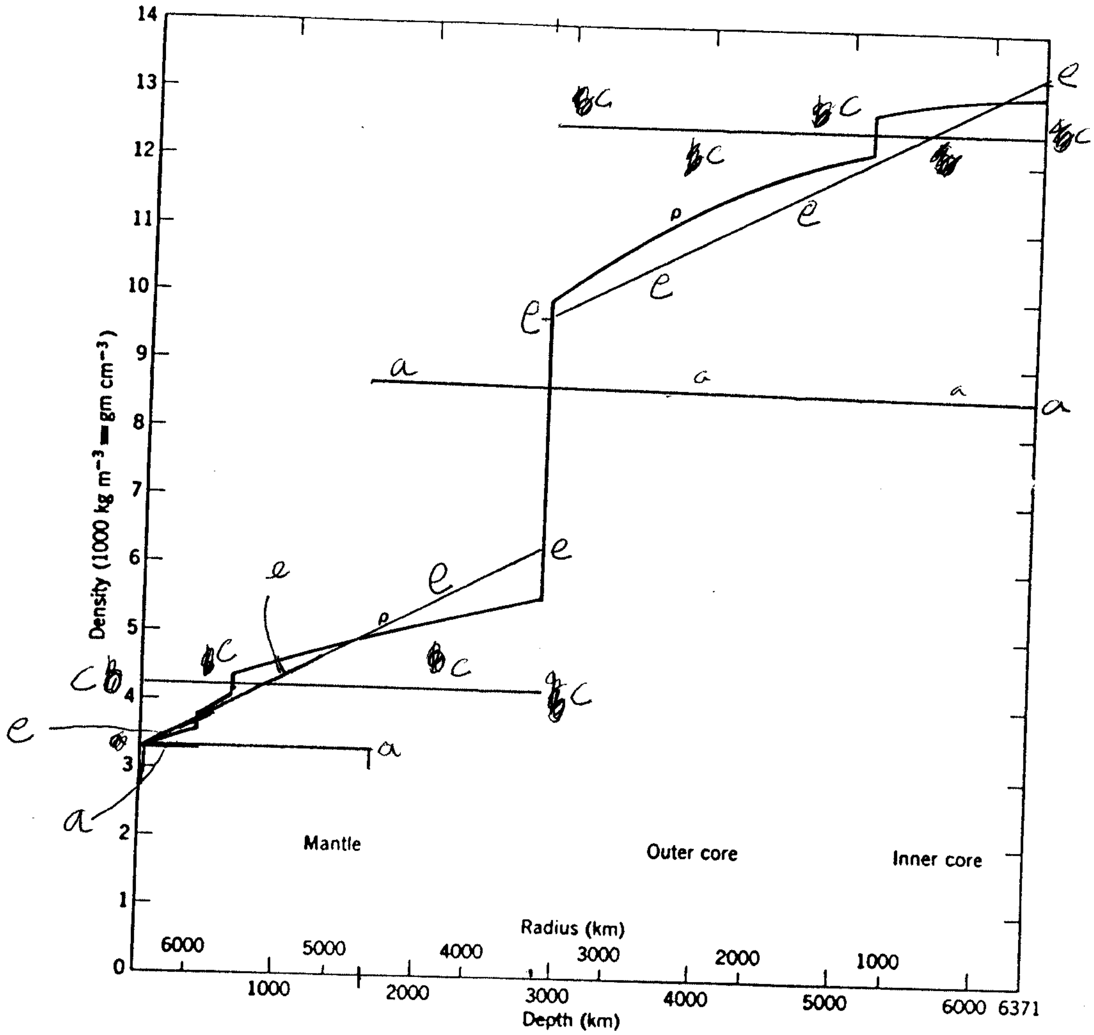
Plugging 6 into 4b,

7 $\begin{cases} 1.265 = (1.060\times10^{8})(1.026\times10^{-8}) + 5.01\times10^{-2} \rho_D \\ 1.774\times10^{-1} = 5.01\times10^{-2} \rho_D \quad ; \quad \boxed{\rho_D = 3.542} \end{cases}$

Plugging 6 into 2a

9 $C_m = 3.3 + (6.37\times10^{8})(1.192\times10^{-3}) = C_m = \boxed{9.836}$

f) Models a, c, and e represent increasingly good fits to the seismically determined density structure of the earth. Model a is not very good - it consistently underestimates density, though the density contrast at the CMB is almost right. Model c is a little better, estimating about the average mantle density but overestimating core density. The density contrast at the CMB is too small. Model e is the best, modeling the mantle very well, while not quite so accurate in the core. Allowing a different density gradient in the core would have helped here. The density contrast at the CMB is too small in this model.



Density profile of Earth model by Dziewonski et al. (1975) (solid line)

model a)

$$r = \begin{cases} 0-4808 \text{ km} : & \rho = 8.5 \text{ g/cm}^3 \\ 4808-6371 \text{ km} : & \rho = 3.3 \text{ g/cm}^3 \end{cases}$$

model c)

$$r = \begin{cases} 0-3500 \text{ km} : & \rho = 12.5 \text{ g/cm}^3 \\ 3500-6371 \text{ km} : & \rho = 4.15 \text{ g/cm}^3 \end{cases}$$

model e)

$$r = \begin{cases} 0-3500 \text{ km} : & \rho = 13.38 - 1.026\times10^{-3} r \\ 3500-6371 \text{ km} : & \rho = 9.836 - 1.026\times10^{-3} r \end{cases}$$

r in km